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A quantum exactly solvable nonlinear oscillator related to the isotonic oscillator

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Abstract

A nonpolynomial one-dimensional quantum potential representing an oscillator, which can be considered as placed in the middle between the harmonic oscillator and the isotonic oscillator (harmonic oscillator with a centripetal barrier), is studied. First the general case, that depends on a parameter a , is considered and then a particular case is studied with great detail. It is proven that it is Schrödinger solvable and then the wavefunctions Ψ_n and the energies E_n of the bound states are explicitly obtained. Finally, it is proven that the solutions determine a family of orthogonal polynomials $\mathcal{P}_n(x)$ related to the Hermite polynomials and such that: (i) every \mathcal{P}_n is a linear combination of three Hermite polynomials and (ii) they are orthogonal with respect to a new measure obtained by modifying the classic Hermite measure.

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1. Introduction

It is well known that, in quantum mechanics, the family of Schrödinger solvable potentials is very restricted and also that exact solvability is a very delicate property (see [1] for a review and [2–11] for some recent papers on this matter). In fact, in most of cases, the addition of a small perturbation to a quantum solvable system breaks this property and leads to potentials that must be analysed by the use of perturbative methods, variational formalisms or numerical techniques. The most interesting and best known system inside this small family

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is the harmonic oscillator whose energy spectrum consists of an infinite set of equidistant energy levels. Many other oscillators, as for example harmonic oscillators perturbed by a term containing a fourth or a sixth power in the coordinate, have been extensively studied but making use of the above-mentioned techniques. Nevertheless, it is known the existence of another oscillator, the so-called isotonic oscillator, that is exactly solvable and is endowed with many interesting properties.

The main aim of this paper is to present a study of a new nonpolynomial one-dimensional quantum potential representing an oscillator, which can be considered as placed in the middle between the harmonic oscillator and the isotonic oscillator. We will prove that it is exactly solvable and that the energy spectrum and the wavefunctions have properties closely related to those characterizing the harmonic oscillator.

In more detail, the plan of the paper is as follows. In section 2, we recall the main characteristics of the isotonic oscillator. In section 3, we present a new potential depending of a parameter a ; first, we study the general case and then we solve the particular case $a^2 = 1/2$ obtaining the wavefunctions and energy spectrum. Section 4 is devoted to analysing a family of orthogonal polynomials and to studying the relation with the Hermite polynomials. Finally, in section 5 we make some final comments.

2. The isotonic oscillator

The following potential:

$$U_{\text{Isot}}(x) = U_0(x) + U_g(x) = \left(\frac{1}{2}\right)\omega^2x^2 + \left(\frac{1}{2}\right)\frac{g}{x^2}, \quad g > 0,$$

representing an harmonic oscillator with a centripetal barrier, is known as the isotonic oscillator [12, 13]. It is important because it is endowed with properties closely related to those of the harmonic oscillator. At the classical level it leads to periodic solutions with the same period (isochronous potential [14]) and at the quantum level it is Schrödinger solvable, the Hamiltonian is factorizable [15] and the energy spectrum is equidistant. Moreover, it is related to supersymmetric quantum mechanics [16]. The two-dimensional version is superintegrable and corresponds to the so-called Smorodinsky–Winternitz system and the centripetal term relates it with the Calogero–Moser system. In addition to all these theoretical properties, this particular nonlinear oscillator is important in quantum optics and in the theory of coherent states [17, 18].

The classical Lagrange equation, that is given by

$$\ddot{x} + \omega^2x - \frac{g}{x^3} = 0,$$

is a particular case of the Pinney–Ermakov equation [19]. It can be exactly solved and the solution is given by

$$x = \frac{1}{\omega A} \sqrt{(\omega^2 A^4 - g) \sin^2(\omega t + \phi) + g},$$

showing explicitly the above-mentioned periodicity. At the quantum level it is convenient to write g in the way $g = m(m + 1)$ (the constant g should be greater than $-1/4$ and m can be any real number but we will always take it as non-negative) so that the Schrödinger equation takes the form

$$\frac{d^2}{dx^2}\Psi - \left[\omega^2x^2 + \frac{m(m+1)}{x^2}\right]\Psi + 2E\Psi = 0,$$

where we assume $\hbar = 1$ for easy of notation. The simplest solution, representing the ground state, is

$$\Psi_0 = N_0 x^{1+m} \exp\left(-\frac{1}{2}\omega x^2\right), \quad E_0 = \left(\frac{3}{2} + m\right)\omega,$$

and all the other wavefunctions and eigenenergies are given by

$$\Psi_{2n} = N_{2n} x^{1+m} P_{2n}(x) \exp\left(-\frac{1}{2}\omega x^2\right), \quad E_{2n} = E_0 + 2n\omega,$$

where $P_{2n}(x)$ is a polynomial of order $2n$ with only even powers of x and N_{2n} denotes the normalization constant. The energy spectrum is equidistant since

$$E_{2n+2} = E_{2n} + 2\omega, \quad n = 0, 1, 2, \dots$$

Nevertheless, the height ΔE of the energy steps is twice that of the simple harmonic oscillator U_0 . In fact, it seems as if half of the levels (those with an odd number of nodes) have disappeared.

3. A new solvable potential

We now turn our attention to the study of the one-dimensional quantum system described by the following potential:

$$U_{0a}(x) = U_0(x) + U_a(x) = \left(\frac{1}{2}\right) \left[\omega^2 x^2 + 2g_a \frac{x^2 - a^2}{(x^2 + a^2)^2} \right], \quad g_a > 0,$$

where a is a positive real parameter. The reason for this particular algebraic expression is that the new additional term can be written as the sum of two centripetal barriers in the complex plane

$$2 \frac{x^2 - a^2}{(x^2 + a^2)^2} = \frac{1}{(x + ia)^2} + \frac{1}{(x - ia)^2},$$

so it is a rational potential with two imaginary poles symmetric with respect the origin. Actually, if g_a remains constant, then when a goes to zero and to ∞ the potential $U_{0a}(x)$ converges to the isotonic and the harmonic oscillator, respectively.

The first important property is that if the coefficient g_a is not arbitrary but given by $g_a = 2\omega a^2(1 + 2\omega a^2)$ then the Schrödinger equation, that takes the form

$$\frac{d^2}{dx^2} \Psi - \left[\omega^2 x^2 + 4\omega a^2(1 + 2\omega a^2) \frac{x^2 - a^2}{(x^2 + a^2)^2} \right] \Psi + 2E\Psi = 0, \quad (1)$$

admits the following solution:

$$\Psi_0 = \frac{N_0}{(a^2 + x^2)^{2\omega a^2}} \exp\left(-\frac{1}{2}\omega x^2\right), \quad E_0 = \frac{1}{2}\omega - (2\omega a)^2. \quad (2)$$

It represents the ground level (Ψ_0 has no nodes) and it clearly shows that when $a \rightarrow 0$ then the corresponding wavefunction and ground-level energy of the linear oscillator is obtained. We also note that the above expression $g_a = 2\omega a^2(1 + 2\omega a^2)$ can alternatively be written as $g_a = m_a(m_a + 1)$ with $m_a = 2\omega a^2$.

In the following, we will focus our study on the particular case $a^2 = 1/2$, $m_a = \omega$ (see figure 1) represented by the equation

$$\frac{d^2}{dx^2} \Psi - \left[x^2 + 8 \frac{2x^2 - 1}{(2x^2 + 1)^2} \right] \Psi + 2E\Psi = 0 \quad (3)$$

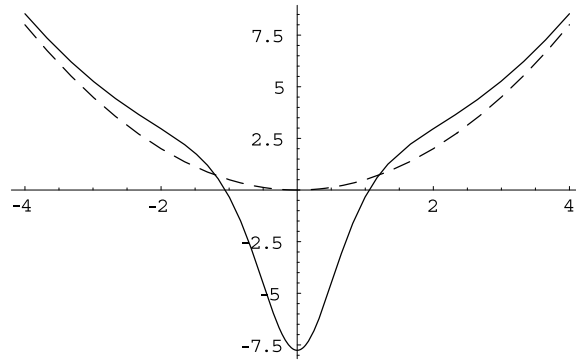


Figure 1. Plot of the potential $U_{0a}(x)$ for $\omega = 1$ as a function of x for $a^2 = 1/2$ (continuous line) together with the plot of the harmonic oscillator (dash line). The main difference lies in the form of the minimum that is much deeper in the $U_{0a}(x)$ case than in the linear case. Nevertheless, for great values of $|x|$ the two functions have rather the same form.

so that Ψ_0 and E_0 become

$$\Psi_0(x) = \frac{N_0}{1 + 2x^2} \exp\left(-\frac{1}{2}x^2\right), \quad E_0 = -\frac{3}{2}, \tag{4}$$

where we have assumed $\omega = 1$ for easy of notation.

As a method for obtaining all the other eigenstates we assume that the functions $\Psi(x)$ can be factorized in the form

$$\Psi(x) = F(x)\Psi_0(x)$$

and, as the lowest energy is $E_0 = -3/2$, it seems appropriate to write the general energy E as follows:

$$E = -\frac{3}{2} + e.$$

Then the Schrödinger equation (3) leads to the following equation for the function $F(x)$:

$$(1 + 2x^2)F'' - 2x(5 + 2x^2)F' + 2e(1 + 2x^2)F = 0. \tag{5}$$

Since the origin $x = 0$ is an ordinary point we expect an analytic solution F with a power-series expansion convergent in the interval $(-R, R)$ with the radius of convergence R given by $R = 1/\sqrt{2}$,

$$F(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \dots$$

that when replaced in (5) leads to

$$2(p_2 + ep_0) + [6p_3 + 2(e - 5)p_1]x + \sum_{m=0}^{\infty} [(m + 4)(m + 3)p_{m+4} + 2[(m + 2)(m - 4) + e]p_{m+2} + 4(e - m)p_m]x^{m+2} = 0. \tag{6}$$

Therefore, we first obtain the following two relations for the first coefficients p_0, p_2 and p_1, p_3 :

$$p_2 + ep_0 = 0, \quad 3p_3 + (e - 5)p_1 = 0,$$

and then the general recursion relation

$$(m + 4)(m + 3)p_{m+4} + 2[(m + 2)(m - 4) + e]p_{m+2} + 4(e - m)p_m = 0, \quad m = 0, 1, 2, \dots$$

Hence the recurrence relation involves three different terms (p_{m+4} depends of both p_{m+2} and p_m) and is constrained by the two first relations that are not included in the general rule. In any case even coefficients are related among themselves and the same is true for odd coefficients. The general solution will be obtained by fixing the values of p_0 and p_1 , i.e. the value $F(0)$ and $F'(0)$. The solution determined by $F(0) = 1, F'(0) = 0$ is even (only contains even powers) while the one determined by $F(0) = 0, F'(0) = 1$ only contains odd powers of x . Actually the expressions of the first coefficients, in terms of p_0 and p_1 , are given by

$$\begin{aligned}
 p_2 &= -ep_0 & p_8 &= \frac{2^4}{8!}(e-50)(e-6)(e-4)ep_0 \\
 p_4 &= \frac{2^2}{4!}(e-10)ep_0 & p_{10} &= -\frac{2^5}{10!}(e-82)(e-8)(e-6)(e-4)ep_0 \\
 p_6 &= -\frac{2^3}{6!}(e-26)(e-4)ep_0 & \dots & \\
 \text{and} & & & \\
 p_3 &= -\frac{2}{3!}(e-5)p_1 & p_9 &= \frac{2^4}{9!}(e-65)(e-7)(e-5)(e-3)p_1 \\
 p_5 &= \frac{2^2}{5!}(e-17)(e-3)p_1 & p_{11} &= -\frac{2^5}{11!}(e-101)(e-9)(e-7)(e-5)(e-3)p_1 \\
 p_7 &= -\frac{2^3}{7!}(e-37)(e-5)(e-3)p_1 & \dots &
 \end{aligned}$$

In the particular case in which e is an even integer number $e = 2k$, with $k = 3, 4, 5, \dots$, all the coefficients $p_{2(k+r)}$ vanish and the series reduces to a polynomial of degree k in powers of $x^2, P_{2k}(x)$. Similarly, when e is an odd integer number $e = 2k + 1$, with $k = 2, 3, 4, \dots$, all the coefficients $p_{2(k+r)+1}$ with $r > 1$ vanish, and the series reduces to a polynomial P_{2k+1} with only odd powers of x . The first polynomial solutions are

$$\begin{aligned}
 P_0 &= 1 & P_3 &= x + \frac{2}{3}x^3 \\
 P_4 &= 1 - 4x^2 - 4x^4 & P_5 &= x - \frac{4}{5}x^5 \\
 P_6 &= 1 - 6x^2 - 4x^4 + \frac{8}{3}x^6 & P_7 &= x - \frac{2}{3}x^3 - \frac{4}{3}x^5 + \frac{8}{21}x^7 \\
 P_8 &= 1 - 8x^2 - \frac{8}{3}x^4 + \frac{32}{5}x^6 - \frac{16}{15}x^8 & P_9 &= x - \frac{4}{3}x^3 - \frac{8}{5}x^5 + \frac{16}{15}x^7 - \frac{16}{135}x^9 \\
 P_{10} &= 1 - 10x^2 + \frac{32}{3}x^6 - \frac{80}{21}x^8 + \frac{32}{105}x^{10}
 \end{aligned}$$

The two important properties are: first, there are no polynomial solutions of degree $k = 1$ and $k = 2$; and second, all of them have two complex conjugate roots, so that P_3 has an unique real zero, P_4 has two real zeros and, in the general case, P_{2k} and P_{2k+1} with $k > 1$ have $2k - 2$ and $2k - 1$ real zeros, respectively.

Another remarkable property is that when the polynomials P_n are expressed as linear combination of Hermite polynomials then only a finite number of terms are different from zero. We have obtained the following relations for the first cases:

$$\begin{aligned}
 P_4 &= -4H_0 - 4H_2 - \frac{1}{4}H_4 & P_3 &= H_1 + \frac{1}{12}H_3 \\
 P_6 &= 3H_2 + H_4 + \frac{1}{24}H_6 & P_5 &= -H_1 - \frac{1}{2}H_3 - \frac{1}{40}H_5 \\
 P_8 &= -\frac{2}{3}H_4 - \frac{2}{15}H_6 - \frac{1}{240}H_8 & P_7 &= \frac{1}{3}H_3 + \frac{1}{12}H_5 + \frac{1}{336}H_7 \\
 P_{10} &= \frac{1}{12}H_6 + \frac{1}{84}H_8 + \frac{1}{3360}H_{10} & P_9 &= -\frac{1}{20}H_5 - \frac{1}{120}H_7 - \frac{1}{4320}H_9.
 \end{aligned}$$

These particular relations clearly suggest that each polynomial P_{2k} can be written as a linear combination of H_{2k} and the two previous even Hermite polynomials, H_{2k-2} and H_{2k-4} ; similarly, the odd polynomial P_{2k+1} appears as a linear combination of only H_{2k+1}, H_{2k-1} and H_{2k-3} .

4. Wavefunctions and orthogonality relations

The differential equation (5) is not in selfadjoint form because, if we denote by a_0 , a_1 and a_2 the three coefficients of the equation, we have

$$a_0 = 1 + 2x^2, \quad a_1 = -2x(5 + 2x^2), \quad \frac{da_0}{dx} \neq a_1.$$

However, use can be made of an integrating factor $\mu(x)$ such that

$$\frac{d}{dx}[\mu(x)a_0(x)] = \mu(x)a_1(x),$$

so that $\mu(x)$ is given by

$$\mu(x) = \left(\frac{1}{a_0}\right) e^{\int (a_1/a_0) dx} = \frac{e^{-x^2}}{(1 + 2x^2)^3},$$

in such a way that (5) becomes

$$\frac{d}{dx} \left[p(x) \frac{dF}{dx} \right] + 2er(x)F = 0,$$

where the two functions $p = p(x)$ and $r = r(x)$ are given by

$$p(x) = \frac{e^{-x^2}}{(1 + 2x^2)^2}, \quad r(x) = \frac{e^{-x^2}}{(1 + 2x^2)^2}.$$

This equation, with the appropriate conditions for the behaviour of the solutions at the end points, constitutes a Sturm–Liouville (S–L) problem, defined in the real line \mathbb{R} . According to this, the eigenfunctions of the S–L problem are orthogonal with respect to the weight function $r = e^{-x^2}/(1 + 2x^2)^2$, and, in particular, the polynomial solutions P_m , $m = 0, 3, 4, \dots$, of the differential equation (5), satisfy the orthogonality conditions

$$\int_{-\infty}^{\infty} P_m(x)P_n(x) \frac{e^{-x^2}}{(1 + 2x^2)^2} dx = 0, \quad m \neq n.$$

Let us now return to equation (5) and suppose for F' the following factorization:

$$F' = (1 + 2x^2)G.$$

Then we arrive after some calculus (we omit the details) to the following equation for the function G :

$$G'' - 2xG' + 2(e - 3)G = 0,$$

that means that the derivative $P'_n(x)$ of the polynomial $P_n(x)$ must satisfy

$$P'_n = (1 + 2x^2)H_{n-3}, \quad e = n, \quad n = 3, 4, 5, \dots$$

(up to a multiplicative constant). At this point we recall the following two properties of the Hermite polynomials:

$$\begin{aligned} \text{(i)} \quad & 2xH_m = H_{m+1} + 2mH_{m-1} \\ \text{(ii)} \quad & H'_m = 2mH_{m-1}. \end{aligned} \tag{7}$$

Then making use of (i) we arrive to

$$P'_n = \frac{1}{2}[H_{n-1} + 4(n - 2)H_{n-3} + 4(n - 3)(n - 4)H_{n-5}],$$

and making use of (ii) and then integrating we obtain

$$P_n = \frac{1}{4n}[H_n + (4n)H_{n-2} + (4n)(n - 3)H_{n-4}].$$

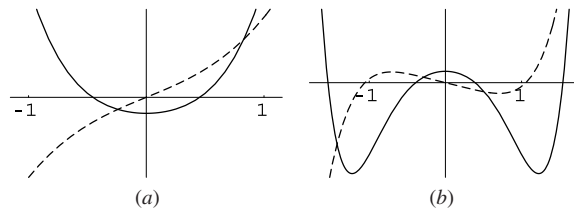


Figure 2. (a) Polynomials \mathcal{P}_3 (dash line) and \mathcal{P}_4 (continuous line). \mathcal{P}_3 has a unique real zero at the origin and \mathcal{P}_4 has two real zeros (symmetric with respect the origin). (b) Polynomials \mathcal{P}_5 (dash line) and \mathcal{P}_6 (continuous line). \mathcal{P}_5 has three real zeros (the origin and two other placed symmetric) and \mathcal{P}_6 has four zeros (two positive and two negative).

It seems convenient to multiply P_n by $4n$ and introduce the new family of polynomials \mathcal{P}_n defined in the form

$$\mathcal{P}_n = H_n + 4nH_{n-2} + 4n(n-3)H_{n-4}, \quad n = 3, 4, 5, \dots, \tag{8}$$

so that the coefficient of H_n (the dominant term in the expression of \mathcal{P}_n) reduces to unity.

Proposition 1. *The following equality holds:*

$$\frac{2n\mathcal{P}_n e^{-x^2}}{(1+2x^2)^2} = -\frac{d}{dx} \left[\frac{H_{n-3}}{1+2x^2} e^{-x^2} \right], \quad n = 3, 4, 5, \dots$$

Proof. This statement is proven just by making the calculus. □

Now we can write

$$\int_{-\infty}^{\infty} \frac{\mathcal{P}_m(x)\mathcal{P}_n(x)}{(1+2x^2)^2} e^{-x^2} dx = -(16nm) \int_{-\infty}^{\infty} \frac{1}{2m} \frac{d}{dx} \left[\frac{H_{m-3}}{1+2x^2} e^{-x^2} \right] \mathcal{P}_n(x) dx$$

and integrating by parts we arrive to

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathcal{P}_m(x)\mathcal{P}_n(x)}{(1+2x^2)^2} e^{-x^2} dx &= (8n) \int_{-\infty}^{\infty} \left[\frac{H_{m-3}}{1+2x^2} e^{-x^2} \right] \mathcal{P}'_n(x) dx \\ &= (8n) \int_{-\infty}^{\infty} H_{m-3} H_{n-3} e^{-x^2} dx \\ &= \delta_{mn} (8n) [2^{n-3} (n-3)! \sqrt{\pi}]. \end{aligned} \tag{9}$$

So, we relate the problem of normalization to the standard problem of Hermite polynomials. Hence the orthogonality conditions for the family $\mathcal{P}_n(x)$ read

$$\int_{-\infty}^{\infty} \mathcal{P}_m(x)\mathcal{P}_n(x)r(x) dx = \delta_{mn} \int_{-\infty}^{\infty} \frac{[\mathcal{P}_n(x)]^2}{(1+2x^2)^2} e^{-x^2} dx = k_n (2^n n! \sqrt{\pi}),$$

where the proportionality constant k_n is given by

$$k_n = \frac{1}{(n-1)(n-2)}.$$

See figures 2(a) and (b) for the plot of some of these polynomials.

If we define the P-Hermite functions Ψ_m by

$$\Psi_m(x) = \frac{\mathcal{P}_m(x)}{(1+2x^2)} e^{-(1/2)x^2}, \quad m = 0, 3, 4, \dots,$$

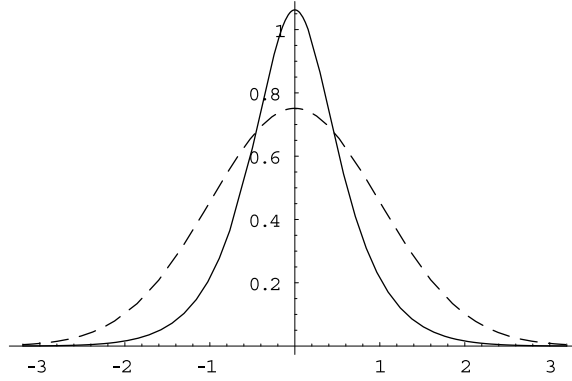


Figure 3. Wavefunction Ψ_0 (continuous line) and wavefunction Φ_0 of the harmonic oscillator (dash line).

then the above property admits the following alternative form:

$$\int_{-\infty}^{\infty} \Psi_m(x)\Psi_n(x) dx = 0, \quad m \neq n.$$

In summary, the eigenfunctions corresponding to the lowest energy levels are

$$\begin{aligned} \Psi_0(x) &= N_0 \frac{\mathcal{P}_0(x)}{(1+2x^2)} e^{-(1/2)x^2}, & E_0 &= -3/2 \\ \Psi_3(x) &= N_3 \frac{\mathcal{P}_3(x)}{(1+2x^2)} e^{-(1/2)x^2}, & E_3 &= 3/2 = -3/2 + 3 \\ \Psi_4(x) &= N_4 \frac{\mathcal{P}_4(x)}{(1+2x^2)} e^{-(1/2)x^2}, & E_4 &= 5/2 = -3/2 + 4 \\ \Psi_5(x) &= N_5 \frac{\mathcal{P}_5(x)}{(1+2x^2)} e^{-(1/2)x^2}, & E_5 &= 7/2 = -3/2 + 5, \end{aligned}$$

where the normalization constant is

$$N_k = \left[\frac{(k-1)(k-2)}{2^k k! \sqrt{\pi}} \right]^{1/2}.$$

The first three wavefunctions, $\Psi_0(x)$, $\Psi_3(x)$, $\Psi_4(x)$, together with the corresponding wavefunctions of the harmonic oscillator $\Phi_0(x)$, $\Phi_1(x)$, $\Phi_2(x)$, are plotted in figures 3–5.

The energy E_0 of the ground state $\Psi_0(x)$ has been singled out of all the other values and moved into the smaller value $E_0 = -3/2$. The rest of the energy spectrum consists, as in the pure harmonic case, of an infinite set of equidistant energy levels

$$E_{n+1} = E_n + 1, \quad n = 3, 4, 5, \dots$$

Let us close this section with two comments on the new family of polynomials we have obtained. First, definition (8) of \mathcal{P}_n as a linear combination of *only three* Hermite polynomials can be considered as a particular case of a situation known as a *special linear combinations of orthogonal polynomials* (see [20, 21] and references therein). Finally, let us mention that taking into account the ‘Rodrigues formula’ for the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d}{dx^n} e^{-x^2}$$

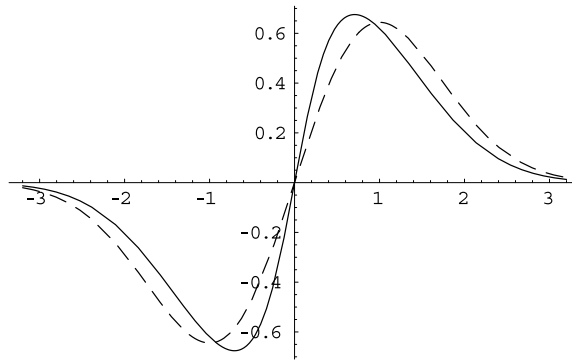


Figure 4. Wavefunction Ψ_3 (continuous line) and wavefunction Φ_1 of the harmonic oscillator (dash line).

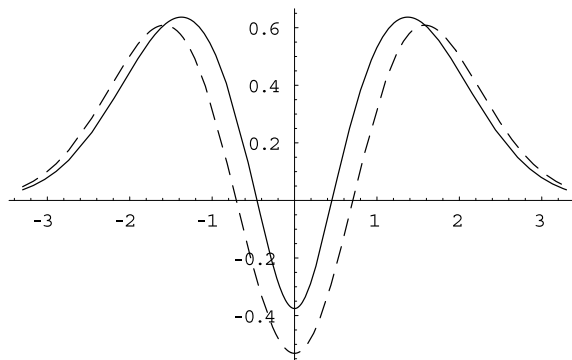


Figure 5. Wavefunction Ψ_4 (continuous line) and wavefunction Φ_2 of the harmonic oscillator (dash line).

we obtain

$$\mathcal{P}_n(x) = (-1)^n e^{x^2} \left[\frac{d^n}{dx^n} + 4n \frac{d^{n-2}}{dx^{n-2}} + 4n(n-3) \frac{d^{n-4}}{dx^{n-4}} \right] e^{-x^2},$$

that must be considered as the ‘Rodrigues formula’ for this new family of orthogonal polynomials.

5. Final comments and outlook

We have proved that the potential $U_{0a}(x)$ can be exactly solved in the particular case $a^2 = 1/2$ and also that it possesses two very remarkable properties. First, the fundamental level Ψ_0 has an energy E_0 that is lower than in the pure harmonic case and, in a sense, is isolated of all the other values. Second, the rest of the energy spectrum is endowed with the equidistance property. Concerning the general case, with an arbitrary value for the parameter a , we have only obtained the expression for the fundamental level ($\Psi_0(a), E_0(a)$). The resolution of the general case remains as an open question that deserves to be studied. Finally, let us also mention that the analysis of this potential using the supersymmetric quantum mechanics as an approach also seems an interesting matter to be studied.

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